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LETTER TO THE EDITOR

Integrals and symmetries: the Bernoulli–Laplace–Lenz vector

K H Mariwalla

Matscience, The Institute of Mathematical Sciences, Madras-600 113, India

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Abstract. The notions of 'independent constants of motion' are clarified emphasising the role of precise definition of the background space. Using a well defined algorithm that relates symmetries of equations of motion of a classical system (Newtonian, Lagrangian or Hamiltonian) to a conserved object in one-to-one fashion, the symmetries of the Coulomb and linear force problems are determined.

Prince and Eliezer (1981) have claimed to have obtained the so-called Lenz vector† from an analysis of the space–time dilation symmetry of the equations of motion for the Coulomb problem. Schafir (1981) has correctly pointed out that, dilation being a single symmetry, it could not give rise to n (components of a vector in n dimensions) conserved objects; moreover if each of the n objects is obtained separately from the same dilation symmetry, there is no insurance that these n objects would constitute components of a vector. Unfortunately, Schafir's paper is itself marred by some popular folklore likely to mislead. The purpose of this note is to give a relatively precise formulation and recall the well defined relation between symmetries and conservation laws (Mariwalla 1975a, b, 1980, 1981) that permit one to obtain the Bernoulli–Laplace–Lenz vector from a well defined invariance transformation of the equations of motion, showing incidentally that dilation symmetry merely gives the energy integral.

A $2n$-dimensional differentiable manifold M^{2n} endowed with a closed, non-degenerate two-form defines a symplectic structure (Whittaker 1961). When the two-form is exact, the manifold can be taken as cotangent bundle T^*V of an n-dimensional manifold V^n with the two-form $\omega = \mathbf{d}A = \mathrm{d}\boldsymbol{p} \wedge \mathrm{d}\boldsymbol{q} = \mathrm{d}p_j \wedge \mathrm{d}q^j$ in the local chart $\{U: q^1 \ldots q^n; p_1 \ldots p_n\}$. The symplectic structure implies that $A = \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{q}$ and ω are respectively relative and absolute integral invariants with respect to symplectic (canonical) transformations (of $R^{2n} \to R^{2n}$). The integral curves of a vector field on this symplectic manifold admit the relative integral invariant A, if and only if these equations ($\dot{\boldsymbol{q}} = \boldsymbol{\xi}, \dot{\boldsymbol{p}} = \boldsymbol{\eta}$) are of the Hamiltonian form ($\boldsymbol{\xi} = \partial H/\partial \boldsymbol{p}, \boldsymbol{\eta} = -\partial H/\partial \boldsymbol{q}$). A time-independent function of q^j, p_j is constant or an integral of the motion when it is in involution (i.e. has vanishing Poisson bracket) with the Hamiltonian function. The Poisson bracket of two such constants of motion is also an integral. In this manner

† Its prehistory involves Gibbs and Wilson (1901), Hamilton (1847), Laplace (1798), Bernoulli (1710), and Hermann (1710a, b) who only gives its magnitude. See for details Volk (1975), Goldstein (1976). A dot denotes differentiation with respect to t, $r = |\boldsymbol{X}| = |\boldsymbol{q}|$; summation conventions of Einstein and vector notation are employed; $\boldsymbol{d}$ is exterior derivative.

one may obtain several such integrals. How many independent integrals exist depends on the precise definition of the term 'independent' and has been a source of confusion. In a $2n$-dimensional manifold there can clearly be no more than $2n$ algebraically independent functions. Hence, in that sense there are at most $2n$ 'essentially new' integrals. This view of independence is perhaps not very useful, as one usually also requires the integrals to be single valued (Landau and Lifshitz 1960). As noted by Wintner (1941), even this criterion is not really effective in singling out 'useful' integrals or weaning out the 'worthless' ones; he introduces the notion of '*isolating*' integrals, for which however, there is no precise mathematical characterisation. An alternative view is that the integrals are independent in the sense of a Lie algebra. Thus for a harmonic oscillator, the symmetry group of the Hamiltonian is $\mathrm{SU}(n)$ giving in all n^2 integrals, including the Hamiltonian function. That this is the maximum follows from the fact that at a point in configuration space there can be at most n linearly independent vectors, involving in all n^2 numbers (Mariwalla 1980). Of these however only $2n$ (including the Hamiltonian) are functionally independent. In practice the determination of functionally independent constants can be a source of confusion on a different account, requiring careful attention to the geometry adopted. We illustrate this for the problem $\ddot{\boldsymbol{X}}+K\boldsymbol{X}=0$. If this is considered as a problem of linear force on a Euclidean background space, its symmetry is that of $\mathrm{GL}(n, R)$, giving rise to n^2 integrals which give in turn the Lie algebra of $\mathrm{GL}(n, R)$ or $\mathrm{U}(n)$ according as $-K \gtrless 0$. But one may also consider this equation to be that of a geodesic on a space of constant (Riemannian) curvature $K \lessgtr 0$. In that case the symmetry of the equations of motion is an (n^2+2n)-parameter group of projective motions on a space of constant curvature. The acceptance of this curved space metric implies the *a priori* use of the energy and angular momentum integrals of the flat background space problem. For instance with the constant curvature metric

$$\dot{\boldsymbol{X}}^2+\varepsilon(\boldsymbol{X}\cdot\dot{\boldsymbol{X}})^2/(1-\varepsilon\boldsymbol{X}^2)=K/\varepsilon m, \qquad \operatorname{sgn}\varepsilon=\operatorname{sgn}K, \tag{1}$$

the equation $m\ddot{\boldsymbol{X}}+K\boldsymbol{X}=0$ is invariant under the isometry

$$\boldsymbol{X}\to\boldsymbol{X}+\boldsymbol{a}[(1-\varepsilon\boldsymbol{X}^2)^{1/2}-\varepsilon\boldsymbol{X}\cdot\boldsymbol{a}f(a)], \tag{2}$$

and gives the conserved vector ($\boldsymbol{l}\equiv\boldsymbol{X}\wedge\boldsymbol{p}$, $\boldsymbol{p}=m\dot{\boldsymbol{X}}$)

$$\Lambda=(\boldsymbol{p}-\varepsilon\boldsymbol{X}\wedge\boldsymbol{l})(1-\varepsilon\boldsymbol{X}^2)^{-1/2}. \tag{3}$$

Since this involves the *a priori* conservation of energy and angular momenta and there is a square root, its expression in terms of flat (background) space integrals presents problems.

Again, let $\boldsymbol{X}=\boldsymbol{Y}+\boldsymbol{A}$, such that

$$\ddot{\boldsymbol{Y}}+K\boldsymbol{Y}+K\boldsymbol{A}=0, \qquad \mathrm{d}\boldsymbol{A}/\mathrm{d}t=0. \tag{4}$$

A solution of these equations corresponds to the Lagrangian

$$L=\dot{\boldsymbol{Y}}^2/2R-KR/2 \qquad R\equiv|\boldsymbol{Y}|.$$

Constants of the motion $\boldsymbol{A}=-\boldsymbol{Y}\cdot\dot{\boldsymbol{Y}}\dot{\boldsymbol{Y}}+\frac{1}{2}(\dot{\boldsymbol{Y}}^2+KR)\boldsymbol{Y}$ and $\boldsymbol{L}=\boldsymbol{Y}\wedge\dot{\boldsymbol{Y}}$ together give the Lie algebra of SO(4) or SO(3, 1) depending on the sign of K. Thus we see that it is *insufficient* to consider the problem of independent constants of a given equation of motion without a complete specification of the geometry under consideration. Incidentally, these examples take us to the heart of the problem of 'inequivalent Lagrangians', showing that much of the confusion in the literature can be traced to the total disregard of the definition of the nature of the background space on which the given physical problem is defined. In other words, given a differential equation in a local coordinate system, one cannot make much of it unless something definite is specified about the background space. Once a background space is specified, a subset of symmetries admitted by this space which leave equations of motion unchanged may be taken as the symmetry group of equations of motion. Then by an algorithm one obtains the corresponding conservation laws, which are in all $\leqslant n^2$ in number if time independent.

The relation between symmetries of equation of motion and conservation laws is by now rather well defined (Mariwalla 1975a, b, 1980, 1981), bringing within its fold all the so-called 'accidental' or 'hidden' symmetries. These symmetries may be considered in Newtonian, Lagrangian or Hamiltonian formulations.

Let $\ddot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}, \dot{\boldsymbol{X}}, t)$ be a Newtonian equation of motion. Consider the infinitesimal change $\boldsymbol{X} \to \boldsymbol{X} + \varepsilon\boldsymbol{\xi}$, $\mathrm{d}t \to \mathrm{d}t(1+\varepsilon\dot{\xi}_0)$ leaving the equations of motion unchanged. One finds after a little algebra the expression

$$\dot{\boldsymbol{X}} \cdot \delta(\ddot{\boldsymbol{X}} - \boldsymbol{F}) = \varepsilon\left(\dot{\Lambda} - \tfrac{1}{2}\boldsymbol{\xi} \wedge \dot{\boldsymbol{X}} \cdot \boldsymbol{\nabla} \wedge \boldsymbol{F} + (\Delta^j \ddot{X}^k - \dot{X}^j \dot{\Delta}^k)\frac{\partial F_j}{\partial \dot{X}^k} + \Delta^j \frac{\partial F_j}{\partial t}\right), \tag{5}$$

$$\boldsymbol{\Delta} = \dot{\boldsymbol{\xi}} - \xi_0 \dot{\boldsymbol{X}}, \qquad \Lambda = \dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{X}} - \ddot{\boldsymbol{X}} \cdot \boldsymbol{\xi} - 2\dot{\xi}_0 \dot{\boldsymbol{X}}^2. \tag{6}$$

If the equation of motion is invariant under this transformation, the left side will be zero; if further $\boldsymbol{F} = -\nabla\phi(r)$, all but $\dot{\Lambda}$ will vanish, giving Λ as constant of the motion†. In particular, for the potential $\phi = \alpha r^{\beta}$, Newton's equations are unchanged under dilation of space–time, and the corresponding conserved object is

$$\Lambda\,(\text{dilation}) = \text{energy}. \tag{7}$$

Because of this, dilation symmetry cannot give rise to yet another integral. In the case of the linear force problem in flat space, the equation of motion is unchanged under linear transformations. For the transformations $X^j \to X^j + \varepsilon^{jk} X_k$, $\varepsilon^{jk} = \varepsilon^{kj}$, we find $\Lambda = \varepsilon^{jk}(p_j p_k + k X_j X_k)$ as the conserved object. For the case of the Coulomb problem $\phi(r) = k/r$, a symmetry transformation of Newton's equation is

$$\boldsymbol{X} \to \boldsymbol{X}(1 + \boldsymbol{C} \cdot \boldsymbol{X})^{-1}, \qquad \mathrm{d}t \to \mathrm{d}t\,(1 + \boldsymbol{C} \cdot \boldsymbol{X})^{-2}. \tag{8}$$

One finds on substitution into the second of equations (6)

$$\Lambda = \boldsymbol{C} \cdot [\boldsymbol{X} \cdot \boldsymbol{pp} - (p^2 + \phi)\boldsymbol{X}], \tag{9}$$

which may be recognised as the Bernoulli–Laplace–Lenz vector. We note that (8) is indeed a point transformation of $R^n \to R^n$ (though not of $R^{n+1} \to R^{n+1}$), 't' being merely a parameter along the curve $(R \to R^n)$ in R^n. In this connection we note that the equation $\boldsymbol{v} \cdot \boldsymbol{\nabla\sigma} \equiv D\boldsymbol{\sigma}/\mathrm{d}t = \alpha\boldsymbol{\sigma}$ defines for each different function α a set of vectors that are parallel relative to a curve $(R \to R^n)$: $X^i = \varphi^i(t)$, in the sense that they have the same direction; thus $\boldsymbol{\Sigma} = \boldsymbol{\sigma} \exp(\int^t \alpha\, \mathrm{d}t)$ gives $D\boldsymbol{\Sigma}/\mathrm{d}t = 0$. If $\boldsymbol{v} = \mathrm{d}\boldsymbol{X}/\mathrm{d}t$ is a tangent to a curve and $\mathrm{d}\boldsymbol{X}/\mathrm{d}T$ is a unit tangent vector $(D(\mathrm{d}\boldsymbol{X}/\mathrm{d}T)/\mathrm{d}T = 0)$ then $\mathrm{d}T = \mathrm{d}t(\exp \int^t \alpha\, \mathrm{d}t)$. This transformation $(\mathrm{d}t \to \mathrm{d}T)$ does not change the nature of the curve or effect a coordinate change, but merely renormalises the magnitude of the vector in the wake of the 'point (coordinate) transformation'. Hence, when the change $\mathrm{d}t \to \mathrm{d}T$ is effected along with the point transformation, it does not change the nature

† When $\Lambda = 0$, and the symmetry is a space–time isometry, $\Lambda = \mathrm{d}\Omega/\mathrm{d}t$ and Ω is a constant of motion.

of the point transformation. We emphasise that the relationship between symmetries and conservation laws enunciated above is independent of any *a priori* appeal to the Lagrangian formulation.

According to D'Alembert's principle, the work of the constraint force on any virtual variation is zero,

$$\int (m\ddot{\boldsymbol{X}}+\nabla\phi)\cdot\boldsymbol{\xi}\,\mathrm{d}t=0. \tag{10}$$

Since one may conceive of a symmetry vector $\boldsymbol{\xi}$ (of the equation of motion) to define constraint on the system, one obtains the conservation laws of linear and angular momenta and energy as arising from isometries (space–time rotations and translations for flat space) of the background space–time. An extension of this principle is obtained by considering its 'lift' to the 'space of tangents':

$$\int (m\ddot{\boldsymbol{X}}+\nabla\phi)\cdot\delta\dot{\boldsymbol{X}}\,\mathrm{d}t=\varepsilon\int (m\ddot{\boldsymbol{X}}+\nabla\phi)\cdot(\dot{\boldsymbol{\xi}}-\dot{\xi}_0\dot{\boldsymbol{X}})\,\mathrm{d}t=0. \tag{11}$$

A little algebra immediately leads to the result that corresponding to the symmetry change $(\boldsymbol{\xi}, \dot{\xi}_0)$ there exists a conserved object

$$\Lambda=\dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{X}}-\boldsymbol{\xi}\cdot\ddot{\boldsymbol{X}}-2\dot{\xi}_0\dot{\boldsymbol{X}}^2, \tag{12}$$

as before (see equation (6)). In as much as D'Alembert's principle is a precursor of Hamilton's principle and expression (10) refers to the first variation δL (when a Lagrangian exists), the expression (11) would refer to the Euler equation of the second variation $\delta^2 L$ and to the Jacobi (variational) equation of the Euler–Lagrange equation (i.e. an equation satisfied by the difference between neighbouring extremals). These statements include a generalisation of Noether's theorem, the detailed treatment of which is well outside the scope of this note.

In the Hamiltonian formulation one envisages an infinitesimal transformation $\boldsymbol{q}\to\boldsymbol{q}+\varepsilon\boldsymbol{\xi}$, $\boldsymbol{p}\to\boldsymbol{p}+\varepsilon\boldsymbol{\mu}$ leaving the Hamiltonian equations unchanged; one obtains

$$\mu_k\frac{\partial^2 H}{\partial p_j\,\partial p_k}+\left(\xi_0\frac{\partial}{\partial t}+\dot{\xi}_0\right)\frac{\partial H}{\partial p_j}-\left(\dot{\xi}^j-\xi^k\frac{\partial^2 H}{\partial q^k\,\partial p_j}\right)=0, \tag{13a}$$

$$\xi^k\frac{\partial^2 H}{\partial q^j\,\partial q^k}+\left(\xi_0\frac{\partial}{\partial t}+\dot{\xi}_0\right)\frac{\partial H}{\partial q^j}+\left(\dot{\mu}_j+\mu_k\frac{\partial^2 H}{\partial p_k\,\partial q^j}\right)=0, \tag{13b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta H=\frac{\partial\delta H}{\partial t}+\varepsilon\left(\dot{\xi}_0-\frac{\partial\xi_p}{\partial t}\right)\frac{\partial H}{\partial t}+\varepsilon\left(\dot{p}_k\frac{\partial\xi^k}{\partial t}-\dot{q}^k\frac{\partial\mu_k}{\partial t}\right). \tag{13c}$$

For $H, \boldsymbol{\xi}, \dot{\xi}_0, \boldsymbol{\mu}$ independent of explicit dependence on t, δH is a constant of the motion. One verifies that the non-canonical transformation with generators

$$\boldsymbol{\xi}=\boldsymbol{c}\cdot\boldsymbol{q}\boldsymbol{q},\qquad \dot{\xi}_0=-2\boldsymbol{c}\cdot\boldsymbol{q},\qquad \boldsymbol{\mu}=\boldsymbol{c}\cdot\boldsymbol{q}\boldsymbol{p}-\boldsymbol{c}\cdot\boldsymbol{p}\boldsymbol{q} \tag{14}$$

for the Coulomb problem yields in $\delta H=\varepsilon\Lambda$ the expression of the Bernoulli–Laplace-Lenz vector. The same result obtains if one instead varies the Euler–Lagrange equations. The infinitesimal changes equation (14) are easily checked to correspond to the finite form, equation (8).

To conclude, we have shown that for every symmetry of an equation of motion of a classical mechanical system (whether in Newtonian, Lagrangian or Hamiltonian formulation) there is a well defined algorithm which gives the related constant of the motion. Thus dilation symmetry gives energy conservation and the symmetry equation (8) for the Coulomb problem gives the Bernoulli–Laplace–Lenz vector. We have also shown that there are at least two possible notions of 'independent constants of motion', namely algebraic or Lie algebraic, and that there is a further complication if one is not careful to define precisely the nature of the background space used.

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